Proceedings of the Jangjeon Mathematical Society
21 (2018), No. 1. pp. 99 – 110
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2018.21.1.99

SINC AND RADIAL BASIS FUNCTIONS FOR SOLUTION OF RAYLEIGH-STOKES PROBLEM WITH FRACTIONAL DERIVATIVES

JALIL RASHIDINIA, ALI PARSA, AND RAHELEH SALEHI

Abstract. In this study we approximate the solution of two dimensional Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivatives. This approximation is based on radial basis functions (RBFs) and the Sinc quadrature rule to approximate the integral part of fractional derivative. The error analysis of the scheme have been studied and discussed. The illustration example verifies the effectiveness of our method and shows that one can obtain accurate results with a small number of basis functions.

2010 Mathematics Subject Classification. 65D05, 65D30, 65MXX, 65M70, 35Q30, 44-XX, 26A33, 41A55.

Keywords and phrases. Rayleigh-Stokes problem, Fractional calculus, Sinc quadrature rule, Double exponential transformation, Radial basis functions.

1. Introduction.

The models of fractional equations have been araised in many fields of science and engineering. Historical and theoretical aspects of fractional calculus were studied in [1, 2, 3] and the large number references there in. There are many text books related to the application of fractional equations, such as Control theory [4], Biology [5], Chemistry [6], Engineering [7], Physics [8], Continuum Mechanics [9] and many other applications [10, 11]. A brief historical introduction to fractional calculus is given in [12].
Here we consider the two dimensional Rayleigh-Stokes problem with fractional derivative for heated generalized second grade fluid

(1)
$$\begin{aligned}\frac{\partial u(x,y,t)}{\partial t} &= {}_0D_t^{1-\gamma}\left[\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}\right] \\ &+\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}+f(x,y,t), \quad 0<t\le T, \quad (x,y)\in\Omega,\end{aligned}$$

with boundary conditions

$$u(x,y,t)=w_1(x,y,t), \qquad (x,y)\in\partial\Omega \tag{2}$$

and initial condition

$$u(x,y,0)=w_2(x,y), \qquad (x,y)\in\Omega. \tag{3}$$

where $\Omega=[a,b]\times[c,d]$, $0<\gamma<1$ and ${}_0D_t^{1-\gamma}u(x,y,t)$ is the Riemann-Liouville fractional derivative of order $1-\gamma$ which is defined by

$${}_0D_t^{1-\gamma}u(x,y,t)=\frac{\partial}{\partial t}{}_0I_t^{\gamma}u(x,y,t), \tag{4}$$

where ${}_0I_t^{\gamma}$ is the fractional integral operator,

$$(5) \qquad {}_0I_t^{\gamma}u(x,y,t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{u(x,y,\tau)}{(t-\tau)^{1-\gamma}} d\tau.$$

The Rayleigh-Stokes problem is a model of non-Newtonian behavior exhibited by certain fluids, the flow characteristics of non-Newtonian viscoelastic fluids through a dual porous medium [13] and the flow analysis of fluids in fractal reservoir with fractional derivative [14].
The Rayleigh-Stokes problem for a heated second grade fluid investigated in [15, 16]. The Fourier sine transform and the Laplace transform used in [17, 18, 19, 20, 21] for solution of the Rayleigh-Stokes problem.
Several methods applied to approximate the solution of problem (1)-(3), such as the explicit and implicit finite difference method [22, 23, 24, 25, 26], the Fourier sine and the Laplace transform [27] and RBF meshless method [28].
The fractional derivatives and fractional integrals are special form of Abel's type of integrals, having weak singularity, for such type problem, the Sinc methods are quite effective. Okayama et al. [29] developed two new Sinc scheme based on single and double exponential transformation for fractional derivatives. They used these methods to solve the linear Fredholm integral equations of the second kind with weakly singular kernel [30], the results are very accurate for large numbers of SE-Sinc collocation points and small numbers of DE-Sinc points. In fact, the authors in [29, 30, 31] took the idea that was presented by Riley [32] to develop the techniques in Sinc methods to approximate the solution of the second kind weakly singular linear Volterra integral equations. Okayama et al. [29] proposed two new approximate formula for Caputo's fractional derivatives of order $0 < \alpha < 1$, based on both SE-Sinc and DE-Sinc methods.
Baumann and Stenger [33] provided a survey of application of Sinc methods to solve fractional integral, fractional derivatives, fractional equations and fractional differential equations.
In this study we approximate the solution of the problem (1)-(3) by using RBF collocation method and double exponential (DE) Sinc quadrature rule. This paper is organized as follows. In section 2, we review the RBFs approximation method. In section 3, some properties of Sinc function is given. In section 4, we develop the collocation method based on Multiquadrics RBF and DE-Sinc quadrature rule. The error analysis of the proposed method is given in section 5. In section 6, an illustrative example is given. Finally some concluding remarks are given in section 7.

2. RBFs approximation method

The radial basis functions (RBFs) are functions that depend on the distance from some center points, that is reducing the higher dimensional space problem to lower dimension [34, 35, 36, 37, 38, 39].

Table 1. Some well-known RBFs

Name	$\varphi(r)$
Thin plate splines(TPS)	$\varphi(r) = r^{2\beta}\log r,\ \beta \in \mathbb{N}$
Multiquadrics(MQ)	$\varphi(r) = (r^2+c^2)^{\frac{1}{2}}$
Inverse multiquadrics(IMQ)	$\varphi(r) = (r^2+c^2)^{\frac{-1}{2}}$
Gaussians(GAU)	$\varphi(r) = e^{\frac{-r^2}{c^2}}$
Odd degree splines	$\varphi(r) = r^{\beta},\ \beta > 0,\ \beta \notin 2\mathbb{N}$

The approximate expansion of $u(\mathbf{x})$ can be obtained by

$$u(\mathbf{x}) = \sum_{i=1}^{N} d_i \varphi(\|\mathbf{x}_i - \mathbf{x}\|_2) = \sum_{i=1}^{N} d_i \varphi_i(r) \tag{6}$$

where $\mathbf{x}_i,\quad i = 1, 2, .., N$ are center points, the $\|.\|_2$ is the Frobenius norm, d_i are unknown coefficients and φ are RBF functions. There are several kinds of RBFs, some of them presented in Table 1, where c is the shape parameter which takes the arbitrary values. The Multiquaric radial basis function was introduced for solution of partial differential equations by Kansa. The exponential convergence of RBF have been studied by [37, 38, 39]. Here we use Multiquadrics basis function.

3. Sinc function

In this section, we review some properties of Sinc function, Sinc interpolation and Sinc quadrature [40, 41, 42, 43, 44] that we need. The Sinc function is defined by

$$Sinc(t) = \begin{cases} \frac{sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}.$$

Let j be an integer and h be a positive number, the jth shifted Sinc function is defined by

$$S(j,h)(t) = Sinc(\frac{t - jh}{h}).$$

Since major of problems are defined over a finite interval (a, b), we need the transformation that maps a finite interval (a, b) to $\mathbb{R}$. Here we use double exponential transformation [31, 43, 44, 45, 46], as follows

$$t = \psi_{a,b}^{DE}(z) = \frac{b-a}{2} tanh(\frac{\pi}{2} sinh(z)) + \frac{b+a}{2},$$

and its inverse function define by

$$z = (\psi_{a,b}^{DE})^{-1}(t) = \phi_{a,b}^{DE}(t) = log\left[\frac{1}{\pi} log(\frac{t-a}{b-t}) + \sqrt{1 + \left(\frac{1}{\pi} log(\frac{t-a}{b-t})\right)^2}\right],$$

that we can define Sinc points as $t_k^{DE} = \psi_{a,b}^{DE}(kh)$.

Definition 3.1. *Let $\mathcal{D}$ be a simple connected domain and $(a,b) \subset \mathcal{D}$ and let $\beta > 0$. The family of all analytic functions on $\mathcal{D}$ denotes $L_\beta(\mathcal{D})$, and for all $z \in \mathcal{D}$ and a positive constant k, $f(z)$ satisfies: $\mid f(z) \mid \leq k \mid ((z-a)(z-b))^\beta \mid$.*

Let $f(t)$ be the analytic function on a strip domain $\mathcal{D}_d = \{z \in \mathbb{C} :\mid Im(z) \mid < d\}$ for some $d > 0$, and should be bounded in some sense. When in corporate with DE transformations, the condition should be considered on the translated domain

$$\psi^{DE}_{a,b}(\mathcal{D}_d) = \left\{ z \in \mathbb{C} : \left| arg\left(\frac{1}{\pi} log(\frac{t-a}{b-t}) + \sqrt{1 + \left(\frac{1}{\pi} log(\frac{t-a}{b-t})\right)^2} \right) \right| < d \right\}.$$

The truncated Sinc quadrature rule can be defined by

$$\int_a^b f(t)dt = h \sum_{j=-M}^{M} f(\psi(jh))\psi'(jh), \tag{7}$$

Following [29], if $(f/\phi'^{DE}) \in L_\beta(\psi^{DE}_{a,b}(\mathcal{D}_d))$ for $0 < d < \frac{\pi}{2}$, then there exist constants $K_1 > 0$ independent of M, such that

(8)

$$\left| \int_a^b f(s)ds - h^{DE} \sum_{k=-M}^{M} f(\psi^{DE}_{a,b}(kh^{DE}))(\psi^{DE}_{a,b})'(kh^{DE}) \right| \leq K_1 exp\left(\frac{-2\pi dM}{log(\frac{2dM}{\beta})} \right),$$

where $h^{DE} = \frac{log(\frac{2dM}{\beta})}{M}$.

4. The collocation method based on radial basis function and double exponential Sinc quadrature rule

In this section we develop our collocation method based on multiquadrics radial basis function for spatial and temporal variables in the equations (1)-(3). The solution of equations (1)-(3) can be approximated by

$$u(x,y,t) = \sum_{i=1}^{N} d_i \varphi_i(r), \tag{9}$$

where

(10)

$$\varphi_i(r) = \sqrt{(x-x_p)^2 + (y-y_q)^2 + (t-t_z)^2 + c^2},\ p,q,z = 1,2,..,n,\ N = n^3,$$

where the step size and grade points of spatial variables and time variable are defined by

$$h_x = \frac{b-a}{n-1}, h_y = \frac{d-c}{n-1}, h_t = \frac{T}{n-1},\ x_p = (p-1)h_x, y_q = (q-1)h_y, t_z = (z-1)h_t.$$

Now we approximate the integral part of fractional derivative in (4) by means of the DE-Sinc approach. By the change of variable $s = \psi^{DE}_{0,t}(\tau)$, first we transform the given interval $(0,t)$ to $\mathbb{R}$, then the integral ${}_aI_t^{\gamma}[g](t)$ for a given function $g(t)$ can be approximated as follows

$$\begin{aligned}{}_0I_t^{\gamma}[g](t) &= \frac{1}{\Gamma(\gamma)}\int_0^t \frac{g(s)}{(t-s)^{1-\gamma}}ds \\ &= \frac{t^{\gamma}}{\Gamma(\gamma)}\int_{-\infty}^{\infty} \frac{\pi cosh(\tau) g(\psi_{0,t}^{DE}(\tau))}{(1+e^{-\pi sinh(\tau)})(1+e^{\pi sinh(\tau)})^{\gamma}}d\tau,\end{aligned} \tag{11}$$

by applying the quadrature rule (7) we have

$$ {}_0I_t^{\gamma}[g](t) \approx \mathcal{I}_M^{DE}[g](t) = \frac{t^{\gamma}}{\Gamma(\gamma)}h\sum_{k=-M}^{M}\frac{\pi cosh(kh)g(\psi_{0,t}^{DE}(kh))}{(1+e^{-\pi sinh(kh)})(1+e^{\pi sinh(kh)})^{\gamma}}, \tag{12}$$

where $h = \frac{log(\frac{2dM}{\beta})}{M}$.

Applying the operator ${}_0I_t^{\gamma}$ defined on (12) and using series (9), we can estimate the fractional derivative of the equation (1) as

$$\begin{aligned}{}_0D_t^{1-\gamma}(\Delta u(x,y,t)) &= \frac{d}{dt}\left({}_0I_t^{\gamma}[\Delta u]\right) \approx \frac{d}{dt}\left(\mathcal{I}_M^{DE}[\Delta u]\right) \\ &= \frac{h}{\Gamma(\gamma)}\frac{d}{dt}\left[t^{\gamma}\sum_{k=-M}^{M}\sum_{i=1}^{N}\frac{\pi cosh(kh)d_i\Delta\varphi_i(r^{kh})}{(1+e^{-\pi sinh(kh)})(1+e^{\pi sinh(kh)})^{\gamma}}\right],\end{aligned} \tag{13}$$

where

$$\Delta\varphi_i(r^{kh}) = \frac{(x-x_l)^2+(y-y_p)^2+2(\psi_{0,t}^{DE}(kh)-t_q)^2+2c^2}{((x-x_l)^2+(y-y_p)^2+(\psi_{0,t}^{DE}(kh)-t_q)^2+c^2)^{\frac{3}{2}}}. \tag{14}$$

Now by substituting (9) and (13) in equation (1) and using collocation points

$$r_j = (x_{p'}, y_{q'}, t_{z'}),\ \ z', p', q' = 1,2,..,n, \tag{15}$$

we have

$$\begin{aligned}\sum_{i=1}^{N} d_i\frac{(t_{z'}-t_z)}{\varphi_i(r_j)} &= \frac{h}{\Gamma(\gamma)}\left[\frac{d}{dt}\left(t^{\gamma}\sum_{k=-M}^{M}\sum_{i=1}^{N}\frac{\pi cosh(kh)d_i\Delta\varphi_i(r^{kh})}{(1+e^{-\pi sinh(kh)})(1+e^{\pi sinh(kh)})^{\gamma}}\right)\right]\Bigg|_{r_j} \\ &+\sum_{i=1}^{N}\Delta\varphi_i(r_j) + f(x_{p'}, y_{q'}, t_{z'}), \\ &p', q' = 2,3,..,n-1,\ z' = 2,3,..,n.\end{aligned} \tag{16}$$

Now for boundary conditions (2) we have

$$\sum_{i=1}^{N} d_i\varphi_i(r_j) = w_1(x_{p'}, y_{q'}, t_{z'}),\quad (x_{p'}, y_{q'}) \in \partial\Omega,\quad z' = 2,3,..,n, \tag{17}$$

also for initial condition (3) we have

$$\begin{aligned}\sum_{i=1}^{N} d_i\varphi_i(r_j^0) = \sum_{i=1}^{N} d_i\sqrt{(x_{p'}-x_p)^2+(y_{q'}-y_q)^2+t_z^2+c^2} = w_2(x_{p'}, y_{q'}), \\ p', q' = 1,2,..,n.\end{aligned} \tag{18}$$

The system (16) associated with (17) and (18) yield the system of N equations and N unknown d_i. By solving this system and substituting the unknown coefficients in (9) we can approximate the solution of equation (1)-

(3).

5. Error analysis

In this section we give an error bound for RBF collocation method that presented in sections 4. The error analysis in this section can be discussed by using ideas in [36] and by assuming that φ is conditionally positive definite of order m. Suppose further that $\Omega \subseteq \mathbb{R}^d$ is bounded and satisfies an interior cone condition. Let $f(x)$ interpolated by φ and satisfied $\left|f^{(l)}(r)\right| \leq l!K_2^l$ for all $r \in [0,\infty]$, where $K_2 > 0$ and fix $\alpha \in N_0^d$, for every $l \in \mathbb{N}, l \geq max\{\mid \alpha \mid, m-1\}$ there exist positive constants h, K_3, K_4, K_5 such that

$$|f(x) - S_{f,X}(x)| \leqslant K_3 e^{-\frac{K_4}{h}}, \tag{19}$$

and

$$|D^\alpha f(x) - D^\alpha S_{f,X}(x)| \leqslant K_5 h^{l-|\alpha|} |f|_{N_\varphi(\Omega)},$$

where N_φ is a Hilbert space corresponding to φ.
Now to prove the next theorem first we need to define the following Sobolev spaces

$$W^{1,2}(\Omega) = H^1(\Omega) = \left\{ w \in L^2(\Omega) : \frac{dw}{dx} \in L^2(\Omega) \right\}.$$

The inner products and norms in $L^2(\Omega)$ are defined as

$$(w,u) = \int_\Omega wu\, d\Omega, \quad \|w\| = (w,w)^{\frac{1}{2}}, \quad \|w\|_1 = (\nabla w, \nabla w)^{\frac{1}{2}},$$

and $H_0^1(\Omega)$ is the space of functions in $H^1(\Omega)$ that vanish at the boundary. The Sobolev weighted norm on the $H_0^1(\Omega)$ space is defined by

$$\|w\|_{H^1} = \left(\int_\Omega \left(|w|^2 + \Theta |\nabla w|^2\right) d\Omega \right)^{\frac{1}{2}} = \left(\|w\|^2 + \Theta \, \|w\|_1^2 \right)^{\frac{1}{2}},$$

where Θ is positive constant.

Theorem 5.1. *The solution of Rayleigh-Stokes problem (1)-(3) has been approximated by $\bar{u}(x,y,t)$, using the collocation method based on RBF. Assume that $u^*(x,y,t)$ is the computed solutions of the arising systems (16)-(18), then the error bound of the RBF collocation method is given by:*

$$|u(x,y,t) - u^*(x,y,t)| \leqslant K_{R_1} e^{-\frac{K_{R_2}}{h}}.$$

Proof 5.1. *At first, we consider the following relation*

$$|u(x,y,t) - u^*(x,y,t)| \leqslant |u(x,y,t) - \bar{u}(x,y,t)| + |\bar{u}(x,y,t) - u^*(x,y,t)| \tag{20}$$

from (19) we have

$$|u(x,y,t) - \bar{u}(x,y,t)| \leqslant K_3 e^{-\frac{K_4}{h}}. \tag{21}$$

To determine the second term of (20), by substituting $\bar{u}(x,y,t)$ and $u^*(x,y,t)$ in equation (1) and subtracting we have

$$E_1(X)\frac{\partial E_2(t)}{\partial t} = {}_0D_t^{1-\gamma} E_2(t)\Delta E_1(X) + E_2(t)\Delta E_1(X) + F(x,y,t), \tag{22}$$

where

$$|\bar{u}(x,y,t)-u^{*}(x,y,t)| = E(x,y,t) = E_1(x,y)E_2(t) = E_1(X)E_2(t),$$

and

$$F(x,y,t) = \left|f(x,y,t)-\bar{f}(x,y,t)\right|.$$

Multiplying both side of equation (22) by $E(X,t) = E_1(X)E_2(t)$ *and integrating over* $\Omega \times [0,T]$*, we obtain*

(23)

$$\begin{aligned}\|E_1\|^2 \int_0^T \tfrac{\partial E_2(t)}{\partial t} E_2(t)dt &= \left({}_0D_t^{1-\gamma}E_2(t), E_2(t)\right)(\Delta E_1(X), E_1(X)) \\ &+\|E_2\|^2 (\Delta E_1(X), E_1(X)) + (F, E_1(X)E_2(t)),\end{aligned}$$

Since $E_2(t) \in H_0^1$ *then* $E_2(0) = E_2(T) = 0$ *and the left hand side of equation (23) is vanished, and also* ${}_0D_t^{1-\gamma}E_2(t) = {}_0^CD_t^{1-\gamma}E_2(t)$ *then we have*

(24)

$$\left({}_0D_t^{1-\gamma}E_2(t), E_2(t)\right) = \int_0^T\int_0^t \frac{E_2(t)(\partial E_2(\tau)/\partial\tau)}{(t-\tau)^{1-\gamma}} d\tau dt \leqslant \frac{T^{\gamma+1}}{\Gamma(\gamma+2)}\|E_2\|_1^2,$$

by substituting (24) in (23) we obtain

(25)

$$\begin{aligned}0 &\leqslant \frac{T^{\gamma+1}}{\Gamma(\gamma+2)}\|E_2\|_1^2(\Delta E_1(X), E_1(X)) + \|E_2\|^2(\Delta E_1(X), E_1(X)) + (F, E_1(X)E_2(t)) \\ &= -\frac{T^{\gamma+1}}{\Gamma(\gamma+2)}\|E_1\|_1^2\|E_2\|_1^2 - \|E_1\|_1^2\|E_2\|^2 + (F, E_1(X)E_2(t)) \qquad ,\end{aligned}$$

and using the Poincare inequality

$$\|E\| \leqslant C\|E\|_1,$$

we obtain

$$\frac{1}{C}\|E_1\|^2\|E_2\|^2 + \frac{T^{\gamma+1}}{\Gamma(\gamma+2)}\|E_1\|_1^2\|E_2\|_1^2 \leqslant (F, E_1(X)E_2(t)),$$

then

$$\begin{aligned}\|E_1\|^2\|E_2\|^2 + \tfrac{CT^{\gamma+1}}{\Gamma(\gamma+2)}\|E_1\|_1^2\|E_2\|_1^2 &\leqslant (CF, E_1(X)E_2(t)) \\ &\leqslant \tfrac{C^2}{2}\|F\|^2 + \tfrac{1}{2}\|E_1\|^2\|E_2\|^2.\end{aligned}$$

Finaly we have

(26) $$\|E\|_{H^1} = (\|E_1\|^2\|E_2\|^2 + \Theta\|E_1\|_1^2\|E_2\|_1^2)^{\frac{1}{2}} \leqslant C\|F\| \leqslant K_6 e^{-\frac{K_7}{h}}.$$

where $\Theta = \frac{2CT^{\gamma+1}}{\Gamma(\gamma+2)}$.
Setting $K_{R_1} = K_3 + K_6$ *and* $K_{R_2} = min\{K_4, K_7\}$ *the proof can be compeleted.*

6. Illustrate Example

The above developed method applied on an example to test the efficiency and accuracy of the purposed method.
We consider the following initial-boundary value problem

$$\begin{aligned}\frac{\partial u(x,y,t)}{\partial t} &= {}_0D_t^{1-\gamma}\left[\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}\right]+\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}\\ &+e^{x+y}\left[(1+\gamma)\,t^{\gamma}-2\frac{\Gamma(2+\gamma)}{\Gamma(1+2\gamma)}t^{2\gamma}-2t^{1+\gamma}\right],\ 0<x,y<1,\ 0<t\le 1\end{aligned}$$

and

$$\begin{aligned} u(0,y,t)&=e^{y}t^{1+\gamma},\quad u(1,y,t)=e^{1+y}t^{1+\gamma},\\ u(x,0,t)&=e^{x}t^{1+\gamma},\quad u(x,1,t)=e^{1+x}t^{1+\gamma},\\ u(x,y,0)&=0,\end{aligned}$$

with the exact solution

$$u(x,y,t)=e^{x+y}t^{1+\gamma}.$$

Collocation method (16) associated with boundaries (17) and (18) is applied on the above example, with $M=20, d=\frac{3.14}{2}, \mu=min\{\gamma,1\}, h=\frac{log(\frac{2dM}{\mu})}{M}$, and also by choosing various values of $h_x=h_y=h_t=\frac{1}{2},\frac{1}{3},...,\frac{1}{8}$, various values of $\gamma=0.15,0.5,0.7,0.8,0.9$ and different values of shape parameter c.
The maximum absolute error in the solution are tabulated in Tables 2 and 3. Where in the tables E_∞ for RBF collocation and DE-Sinc quadrature method define as

$$E_\infty=\max_{1\leqslant p\leqslant n}\max_{1\leqslant q\leqslant n}\max_{1\leqslant z\leqslant n}\left\{|u(x_p,y_q,t_z)-u^*(x_p,y_q,t_z)|\right\},$$

where $u^*(x,y,t)$ is approximation solution of $u(x,y,t)$.

Table 2. Error of the RBF collocation and DE-Sinc quadrature method with $M=20$

	$\gamma=0.15$		$\gamma=0.5$	
$h_x=h_y=h_t$	E_∞	c	E_∞	c
$\frac{1}{2}$	6.3897×10^{-4}	3.5	1.9874×10^{-4}	2.9
$\frac{1}{3}$	8.9968×10^{-4}	4.2	9.7772×10^{-4}	2.5
$\frac{1}{4}$	4.8106×10^{-4}	2	1.9155×10^{-4}	2.2
$\frac{1}{5}$	1.5590×10^{-4}	2	4.6520×10^{-4}	2
$\frac{1}{6}$	8.0221×10^{-5}	1.5	3.8226×10^{-5}	3.1
$\frac{1}{7}$	3.9522×10^{-5}	1.5	8.5025×10^{-5}	3
$\frac{1}{8}$	1.5428×10^{-5}	2	2.6742×10^{-5}	2.8

Tables 2 and 3 show that by using the method based on RBF and DE-Sinc method in (16)- (18) with few number of basis functions (small values of N and M), one can obtain good results.

TABLE 3. Error of the RBF collocation and DE-Sinc quadrature method with $M = 20$

	$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
$h_x = h_y = h_t$	E_∞	c	E_∞	c	E_∞	c
$\frac{1}{2}$	3.2564×10^{-3}	2.8	9.2532×10^{-3}	2.8	1.9387×10^{-2}	3
$\frac{1}{3}$	3.2400×10^{-3}	1.5	7.6902×10^{-3}	1.5	6.8439×10^{-3}	1.5
$\frac{1}{4}$	8.6942×10^{-4}	2	4.2834×10^{-3}	2.9	9.7195×10^{-3}	2.5
$\frac{1}{5}$	7.4627×10^{-4}	1.5	2.2732×10^{-4}	3	9.5185×10^{-3}	2.5
$\frac{1}{6}$	4.5463×10^{-4}	2.5	5.7210×10^{-4}	2	4.5345×10^{-3}	2
$\frac{1}{7}$	9.8060×10^{-5}	3.1	6.7349×10^{-5}	2.2	2.4684×10^{-4}	2
$\frac{1}{8}$	3.9089×10^{-4}	2	3.8371×10^{-5}	1.8	1.7207×10^{-4}	2

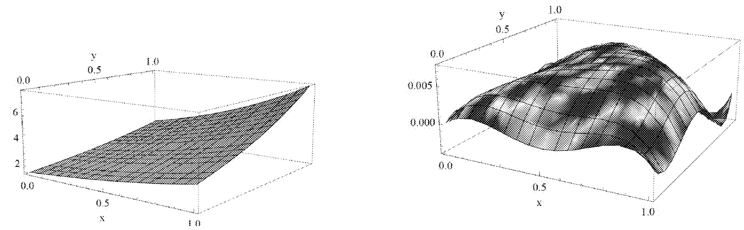

FIGURE 1. Graphs of approximate solution (left panel) and resulting error (right panel) using RBF collocation and DE-Sinc quadrature method at $t = 1$ with $\gamma = 0.7, h_x = h_y = h_t = \frac{1}{3}$ and $c = 2.5$.

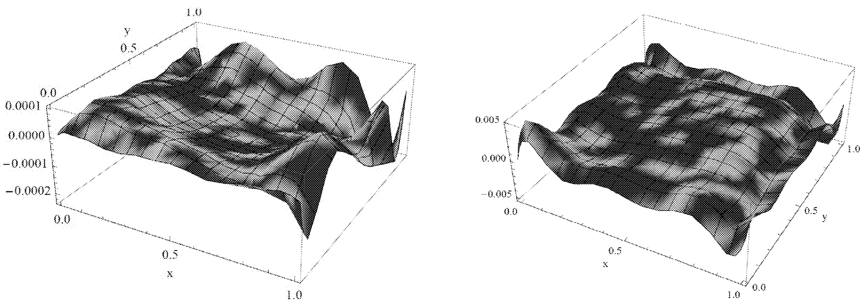

FIGURE 2. Graphs of resulting error using RBF collocation and DE-Sinc quadrature method with $\gamma = 0.15, h_x = h_y = h_t = \frac{1}{5}$ and $c = 1.5$ (left), and with $\gamma = 0.9, h_x = h_y = h_t = \frac{1}{6}$ and $c = 1.5$ (right).

Fig. 1 shows the graphs of approximate solution and resulting error in the test problem using RBF collocation and DE-Sinc quadrature method at

$t = 1$ with $\gamma = 0.7, h_x = h_y = h_t = \frac{1}{3}$ and $c = 2.5$. Fig. 2 shows the graphs of resulting error using RBF collocation and DE-Sinc quadrature method at $t = 1$ with $\gamma = 0.15, h_x = h_y = h_t = \frac{1}{5}$ and $c = 1.5$ (left), and also with $\gamma = 0.9, h_x = h_y = h_t = \frac{1}{6}$ and $c = 1.5$ (right).

7. Conclusion

Our presented method are capable to approximate the solution of the two dimensional Rayleigh-Stokes problem with fractional derivative for heated generalized second grade fluid using combination of Sinc and RBF method. This method is applicable and efficient and can be used with few number of basis functions. Due to the exponentially convergence nature of the method, one can get the considerable good results with small error. The illustrated example shows the efficiency and accuracy of presented method.

References

[1] K. B. Oldham and J. Spanie, *The Fractional Calculus*, Academic Press, New York (1974).

[2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York (1993).

[3] A. A. Kilbas, H. M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam (2006).

[4] S. Das, *Functional Fractional Calculus for System Identification and Controls*, Springer-Verlag Berlin Heidelberg, Berlin (2008).

[5] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House, Connecticut (2006).

[6] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego (1999).

[7] J. Sabatier, O. P. Agrawal and J. A. T. Machado, *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht (2007).

[8] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore (2000).

[9] A. Carpinteri and F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag, Wien (1997).

[10] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer, Heidelberg (2010).

[11] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, Imperial College Press, London, Hackensack NJ (2010).

[12] L. Debnath, A brief historical introduction to fractional calculus, International Journal of Mathematical Education in Science and Technology, 35:4 (2004) 487–501.

[13] L. Shan, D. Tong, L. Xue, Unsteady flow of non-Newtonian visco-elastic fluid in dual-porosity media with the fractional derivative , Journal of Hydrodynamics 21 (2009) 705–713.

[14] J. Tian and D. Tong, The flow analysis of fluids in fractal reservoir with the fractional derivative, Journal of Hydrodynamics, 18 (2006) 287–293.

[15] C. Fetecau and Corina Fetecau, The Rayleigh-Stokes problem for heated second grade fuids ,International Journal of Non-Linear Mechanics, 37 (2002) 1011–1015.

[16] J. Zierep and C. Fetecau, Energetic balance for the Rayleigh-Stokes problem of a second grade fluid,International Journal of Engineering Science 45 (2007) 155–162.

[17] F. Shena, W. Tana, Y. Zhaoc and T. Masuokad, The Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivative model, Nonlinear Analysis: RealWorld Applications, 7 (2006) 1072 – 1080.

[18] C. Fetecu and J. Zierep, On a class of exact solutions of the equations of motion of a second grade fluid, Acta Mechanica, 150 (2001) 135–138 .

[19] M. Khan, The Rayleigh-Stokes problem for an edge in a viscoelastic fluid with a fractional derivative model, Nonlinear Analysis: Real World Applications, 10 (2009) 3190–3195.
[20] C. Fetecau, M. Jamil, C. Fetecau and D. Vieru, The Rayleigh-Stokes problem for an edge in a generalized Oldroyd-B fuid, Z. angew. Math. Phys., 60 (2009) 921–933.
[21] T. O. Salim and A. El-Kahlout, Solution of Fractional Order Rayleigh-Stokes Equations,Adv. Theor. Appl. Mech., 5 (2008) 241 – 254.
[22] Chang-Ming Chen a, F. Liu and V. Anh, Numerical analysis of the Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivatives,Applied Mathematics and Computation, 204 (2008) 340–351.
[23] Ping-hui ZHUANG and Qing-xia LIU, Numerical method of Rayleigh-Stokes problem for heated generalized second grade fluid with fractional derivative, Appl. Math. Mech. -Engl. Ed. 30(12) (2009) 1533–1546.
[24] C. Fetecau and J. Zierep, The Rayleigh-Stokes problem for a Maxwell fluid, Z. angew. Math. Phys., 54 (2003) 1086–1093.
[25] C. Wu, Numerical solution for Stokes first problem for a heated generalized second grade fluid with fractional derivative, Applied Numerical Mathematics, 59 (2009) 2571–2583.
[26] C.M Chen, F. Liu and V. Anh, A Fourier method and an extrapolation technique for Stokes first problem for a heated generalized second grade fluid with fractional derivative, Journal of Computational and Applied Mathematics, 223 (2009) 777–789.
[27] C. Xue, J. Nie ,*Exact solutions of the RayleighStokes problem for a heated generalized second grade fluid in a porous half-space*, Applied Mathematical Modelling, 33 (2009) 524–531.
[28] A. Mohebbi, M. Abbaszadeh and M. Dehghan, Compact finite difference scheme and RBF meshless approach for solving 2D Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivatives, Comput. Methods Appl. Mech. Engrg. (2013).
[29] T. Okayama, T. Matsuo and M. Sugihara, Approximate Formulae for Fractional Derivatives by Means of Sinc Methods, Journal of Concrete and Applicable Mathematics,8 (2010) p.470.
[30] T. Okayama, T. Matsuo and M. Sugihara, Sinc-collocation methods for weakly singular Fredholm integral equations of the second kind, Journal of Computational and Applied Mathematics, 234 (2010) 1211–1227.
[31] Gholam-Ali Zakeri and Mitra Navab, Sinc collocation approximation of non-smooth solution of a nonlinear weakly singular Volterra integral equation, Journal of Computational Physics, 229 (2010) 6548–6557.
[32] B. V. Riley, The numerical solution of Volterra integral equations with nonsmooth solutions based on sinc approximation, Applied Numerical Mathematics, 9 (1992) 249–257.
[33] G. Baumann and F. Stenger, Fractional calculus and Sinc methods, Fractional Calculus and Applied Analysis, 14 (2011) 568–622.
[34] G. E. Fasshouer,*Mesh free approximation methods with MATLAB*, USA, World Scientific (2007).
[35] J. Li and Y.Chen, *Computational Partial Differential Equations Using MATLAB*, CRC Press, Boca Raton, (2008).
[36] H. Wenlland, *Scattered Data Approximation*, Cambridg University Press, New York (2005).
[37] Al. Fedoseyer, M. J. Friedman and E. J. Kansa, Improved multiquadrics method for elliptic partial differential equations via PDE collocation on the boundary, Comput. Math. Appl., 43 (2002) 439-455.
[38] B. Fornberg, G. Wright and E. Larsson, Some observation regarding interpolants in the limit of flat radial basis functions, Comput. Math. Appl., 47 (2004) 37-55.
[39] J. Yoon, Spectral approximation orders of radial basis function interpolation on the Sobolov space, SIAM J. Math. Anal., 33 (1999) 946-958.
[40] F. Stenger, Numerical Methods Based on Sinc and Analytic Functions, Springer-Verlag, New York (1993).

[41] F. Stenger, Handbook of Sinc Numerical Methods, CRC Press, Boca Raton, (2011).
[42] J. Lund and K.L. Bowers, Sinc method for quadrature and differential equations, *SIAM*, (1992).
[43] K. Tanaka, M. Sugihara and K. Murota, Function Classes for Successful DE-Sinc Approximations, Math. Comput., 78 (2009) 1553–1571.
[44] K. Tanaka, M. Sugihara, K. Murota and M. Mori, Function classes for double exponential integration formulas, Numerische Mathematik, 111 (2009) 631–655.
[45] M. Sugihara and T. Matsuo, Recent developments of the Sinc numerical methods, Journal of Computational and Applied Mathematics, 164/165 (2004) 673–689.
[46] M. Mori and M. Sugihara, The double-exponential transformation in numerical analysis, Journal of Computational and Applied Mathematics, 127 (2001) 287–296.

School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 168613114, Iran
E-mail address: rashidinia@iust.ac.ir

School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 168613114, Iran
E-mail address: aliparsa@alumnimail.iust.ac.ir

School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 168613114, Iran
E-mail address: rh_salehi@yahoo.com